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A Mathematical Description of the Concepts of Theoretical Plate and Transfer Unit

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Summary

For reasons of simplicity the concepts of theoretical plate and transfer unit are frequently used in books and papers dealing with countercurrent separation processes. The physical validity of these two concepts cannot be deduced from their definitions. Although the two concepts have been discussed many times, an acceptable relation between the two concepts fails. Moreover, the regions of applicability of the concepts cannot be well indicated. In this paper the concepts of theoretical plate and transfer unit have been related with the column equation. The equations obtained show that both the theoretical plate and the transfer unit are satisfactory units of column length when the compounds are difficult to separate or when the concentration of one component is small with respect to the other. When dealing with exchange systems showing an equilibrium constant deviating from unity, the regions of applicability of the theoretical plate and of the transfer unit may be different. Depending on the parameters of the column equation, the theoretical plate may be preferable to the transfer unit or vice versa. The assumption of many authors that the concept of the transfer unit is more satisfactory than the concept of the theoretical plate when dealing with packed columns is not confirmed by the calculations presented.

INTRODUCTION

In books and papers dealing with countercurrent processes, the concepts of theoretical plate and transfer unit are frequently used for expressing the efficiency of a separation column in numerical terms. Furthermore, these concepts are applied to indicate the difficulty of a distinct separation. The physical validity of these two concepts cannot be deduced from their definitions. Although the two concepts have been discussed in the literature many times, an acceptable relation between the two concepts fails. Moreover, the regions of applicability of the concepts cannot be well indicated.

For a large number of separation methods, the multiplication of the elementary effect by means of a column or a cascade may be described by one basic differential equation, the column equation, as indicated by Cohen (*1*). The authors will show that the desired relation between the concepts of theoretical plate and transfer unit can be derived from this column equation.

In Cohen's treatment of two-phase separation processes like distillation and chemical exchange, the axial dispersion effects have been neglected. Unfortunately, the axial mixing may be rather large in countercurrent processes, especially when packed columns are applied. Therefore, Cohen's equation is less suitable for our purpose.

It is preferable to start with the column equation in which the axial dispersion effects are included, as presented elsewhere (*2*). This column equation has been derived from a mass balance that is maintained for the entire cross section of the column, represented by the equation of continuity (*3*). In a countercurrent column a separation can be obtained only when mass transfer occurs through the interface.

The interfacial mass transfer (determining the values of the concentrations and depending on them) appears in the equation representing the mass balance of one phase. By substituting the mass balance of one phase into the mass balance for the entire cross section of the column, the column equation for countercurrent processes has been obtained. Previously (*2*), a simplified equation for the interfacial mass transfer has been applied. As a consequence, the application of the column equation has been limited to systems of closely related compounds.

Because of our purpose, a more general equation for the interfacial mass transfer is presented in this paper first. Introduction of this relation into the mass balance of two phases yields a rather general column equation that is applicable to a wide variety of systems. On the basis of this column equation, a discussion of the concepts of theoretical plate and transfer unit will be given.

MASS BALANCE

In the following discussion an exchange column connected with one reflux section is considered. This column is represented by Fig. 1. The reflux section is at the bottom of the column.

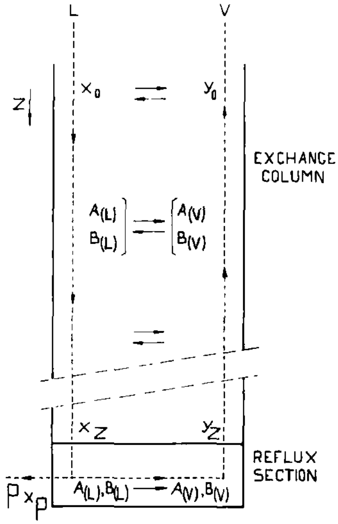

FIG. 1. Separation column considered.

It is assumed that the mixture to be separated consists of two compounds *A* and *B* and that the exchange reaction satisfies the equation:

$$A_V + B_L \rightleftharpoons A_L + B_V$$

L and *V* refer to the descending and ascending phases, respectively. The exchange column is supposed to be a packed one. A packed column may be regarded as an assembly of narrow cross-linked channels. For the sake of simplicity, only one of these channels will be dealt with initially.

The column equation will be derived for a long circular channel with radius r_2. In this channel two phases are moving in countercurrent. Between $r = 0$ and $r = r_1$, one phase is flowing downward; between $r = r_1$ and $r = r_2$ the second phase is transported upward. The interface of the two phases is at $r = r_1$.

The mass balance of component *A* in an infinitesimal section of the column with respect to *z* and *t* may be represented by

$$2\pi \frac{\delta}{\delta z} \int_0^{r_2} \left\{ v_z c_A - D \frac{\delta c_A}{\delta z} \right\} r\, dr = -2\pi \int_0^{r_2} \frac{\delta c_A}{\delta t} r\, dr \qquad (1)$$

where *z* is the coordinate in the longitudinal direction (centimeters) that *z* increases from, *z* is zero at the inlet up to $z = Z$ at the beginning of the reflux section, t is time (seconds), D is the axial dispersion coefficient (square centimeters per second), and c_A is the concentration of component A (moles per milliliter).

The concentration and flow profiles in a channel of a packed column cannot be described. Therefore parameters indicating the total flow streams of and the average concentrations in the phases will be used in this treatment.

Equation (1) will be transformed to Eq. (4) using the following boundary conditions and definitions:

$$2\pi \int_0^{r_2} v_z(c_A + c_B) r \, dr = P \tag{2}$$

$$2\pi \int_0^{r_1} v_z(c_A + c_B) r \, dr = L \tag{2a}$$

$$2\pi \int_{r_1}^{r_2} v_z(c_A + c_B) r \, dr = -V \tag{2b}$$

where P is the product flow (moles per second) and L and V are the flow rate in the descending and ascending phases (moles per second).

Three different fraction parameters will be introduced, viz., the flow rate fraction, the dispersion rate fraction, and the fractional holdup variation.

The flow rate fraction of component A in the two phases may be defined according to:

$$\frac{2\pi \int_0^{r_1} v_z c_A r \, dr}{2\pi \int_0^{r_1} v_z(c_A + c_B) r \, dr} = x \tag{3}$$

$$\frac{2\pi \int_{r_1}^{r_2} v_z c_A r \, dr}{2\pi \int_{r_1}^{r_2} v_z(c_A + c_B) r \, dr} = y \tag{3a}$$

where x and y are the flow rate fractions of component A in the descending and ascending phases.

Furthermore:

$$2\pi \int_0^{r_2} v_z c_A r \, dr = Lx - Vy = Px_P \tag{2c}$$

where x_P is the flow rate fraction of component A in the product flow.

When the product flow is constant,

$$\frac{\delta x}{\delta z} = \frac{V}{L}\frac{\delta y}{\delta z} \qquad \frac{\delta x}{\delta t} = \frac{V}{L}\frac{\delta y}{\delta t} \tag{2d}$$

The dispersion rate fraction of component A may be represented by

$$2\pi \int_0^{r_2} D \frac{\delta c_A}{\delta z} r\,dr = \frac{\delta \bar{x}_D}{\delta z} \{\pi r_1^2 c_L D_L\} \tag{3b}$$

$$2\pi \int_{r_1}^{r_2} D \frac{\delta c_A}{\delta z} r\,dr = \frac{\delta \bar{y}_D}{\delta z} \{\pi (r_2^2 - r_1^2) c_V D_V\} \tag{3c}$$

where $\bar{x}_D$ and $\bar{y}_D$ are the dispersion rate fractions of component A in the descending and ascending phases, c_L and c_V the sum of the concentrations of the components A and B in the descending and ascending phases (moles per milliliter), and D_L and D_V the axial dispersion coefficients in the descending and ascending phases (square centimeters per second).

The fractional holdup variations of component A are given by

$$2\pi \int_0^{r_1} \frac{\delta c_A}{\delta t} r\,dr = \frac{\delta \bar{x}_t}{\delta t} c_L \pi r_1^2 \tag{3d}$$

$$2\pi \int_{r_1}^{r_2} \frac{\delta c_A}{\delta t} r\,dr = \frac{\delta \bar{y}_t}{\delta t} c_V \pi (r_2^2 - r_1^2) \tag{3e}$$

where $\bar{x}_t$ and $\bar{y}_t$ are the fractional holdup variations of component A in the descending and ascending phases.

The various fraction parameters may be easily related to each other, as shown previously (2).

Finally, it is assumed that L, V, c_V, c_L, D_V and D_L are independent of z throughout the column.

Introduction of the definitions and boundary conditions given above and of the relations between the fraction parameters derived in the previous paper (2) into Eq. (1) leads to

$$\frac{\delta}{\delta z}\{Px + V(x - y)\} = \frac{\delta}{\delta z}\left[\{c_2^1 + c_2''x\}\frac{\delta x}{\delta z}\right] - \frac{\delta}{\delta t}\left[\{c_6^1 + c_6''x\}\frac{\delta x}{\delta t}\right] \tag{4}$$

where

$$c_2^1 = [\pi r_1^2 c_L D_L\{1 + b_L f_2\} + \pi (r_2^2 - r_1^2) c_V D_V\{(L/V) - b_V q f_2\}]$$

$$c_2'' = [-\pi r_1^2 c_L D_L b_L f_1 + \pi(r_2^2 - r_1^2) c_V D_V b_V q f_1]$$

$$c_6^1 = [\pi r_1^2 c_L \{1 + b_{L,t} f_2\} + \pi(r_2^2 - r_1^2) c_V \{(L/V) - b_{V,t} q f_2\}]$$

$$c_6'' = [-\pi r_1^2 c_L b_{L,t} f_1 + \pi(r_2^2 - r_1^2) c_V b_{V,t} q f_1]$$

where b_L, b_V, $b_{L,t}$, $b_{V,t}$, f_1, and f_2 are the parameters obtained from relations between the various fraction parameters [presented in the previous paper (2)].

When the contribution of the interfacial mass transfer to the concentration distribution is introduced into the mass balance according to Eq. (4), the column equation is obtained. An expression for the interfacial mass transfer will be derived in the following section. The column equation will be discussed later.

INTERFACIAL MASS TRANSFER

The equilibrium condition at the interface of an exchange reaction according to $A_V + B_L \rightleftharpoons A_L + B_V$ may be described by

$$K = \frac{x^1(1 - y^1)}{y^1(1 - x^1)} \tag{5}$$

where x^1 and y^1 are the flow rate fractions of component A at the boundary and K is the equilibrium constant†; it is assumed that $K > 1$.

The mass transfer of component A through the interface may be represented by

$$F_A = k_L(x^1 - x) = -k_V(y^1 - y) = k_V q(x^1 - x) \tag{6}$$

where F_A is the net interfacial mass transfer of component A in a column of unit length (moles per second-centimeter), k_L and k_V are coefficients of mass transfer in the descending and ascending phases (moles per second-centimeter), x and y are the flow rate fractions of component A in the descending and ascending phases, and $q = k_L/k_V = (y - y^1)/(x^1 - x)$.

As shown previously (2), Eq. (5) leads to a rather complicated general expression for $x^1 - x$. However, a simple and very satisfactory approximation for $x^1 - x$ may be deduced graphically from

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† When considering a distillation column with a reflux section at the top, x refers to the composition of the vapor phase and y to that of the liquid phase. The equilibrium constant K will then be equal to the relative volatility α.

Fig. 2. In Fig. 2 two curves have been represented, the equilibrium curve and the operating line. The equilibrium curve representing the relation between x^1 and y^1 obtains to Eq. (5). The operating line representing the relation between x and y in a cross section of the column obtains to the mass balance according to $Lx - Vy = Px_P$. It will be clear that the operating line represented in Fig. 2 (being the diagonal of the diagram) is the operating line at total reflux.

Moreover, the compositions x^* and y^* have been indicated in Fig. 2. They do not exist physically at the particular positions under consideration, but they represent compositions of the descending and ascending phases, which would be in equilibrium with the bulk compositions of the ascending and descending phases at those points. In contrast with x^1 and y^1, x^* and y^* may be simply related with x and y according to

$$x^* = \frac{Ky}{1 + (K-1)y} \tag{7}$$

and

$$y^* = \frac{x}{K - (K-1)x} \tag{7a}$$

From Fig. 2 and Eqs. (7) and (7a), the following may be deduced:

$$\begin{aligned}(x^1 - x) &= \frac{(x^* - x)}{\{1 + q(x^* - x)/(y - y^*)\}} \\ &= \frac{(K-1)y(1-x) + (y-x)}{1 + Kq + (K-1)(y - qx)}\end{aligned} \tag{8}$$

It may be verified, e.q., by graphical constructions, that this approximation for $x^1 - x$ leads to only small errors for values of the equilibrium constant up to 3.

It may be noted, that when $q \ll 1$, $(x^1 - x) \simeq (x^* - x)$; and when $q \gg 1$, $q(x^1 - x) = (y - y^1) \simeq (y - y^*)$.

Combination of Eqs. (6) and (8) gives an expression for the interfacial mass transfer. Introduction of this equation into the mass balance of one phase leads to

$$F_A = k_L \left\{\frac{(K-1)y(1-x) + (y-x)}{1 + Kq + (K-1)(y - qx)}\right\} = (L + L^*)\frac{\delta x}{\delta z} + c_{\theta,L}\frac{\delta x}{\delta t} \tag{9}$$

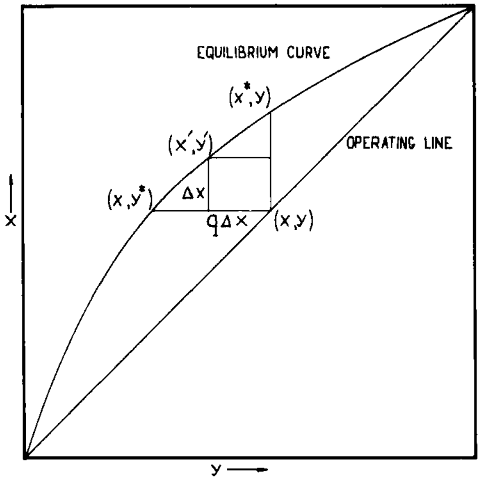

FIG. 2. Graphical relation between x^1 and x.

where L^* stands for corrections due to the axial dispersion and $c_{6,L} = \pi r_1^2 c_L\{1 + b_{L,t}f_2 - x b_{L,t}f_1\}$.

By introducing Eq. (9) into the mass balance for the entire cross section of the column, the column equation may be obtained. This column equation will be discussed in the following section.

COLUMN EQUATION

By deducing a relation for the factor $x - y$ from Eq. (9) and by substituting this relation into Eq. (4), the column equation is obtained. For the steady state of the column, the column equation may be reduced to

$$(c_5^1 + x c_5'')(dx/dz) = c_1 x(1 - x) + Px\{1 + x_p(K - 1)\} - PKx_P \quad (10)$$

where $c_1 = L(K - 1)$, $c_5^1 = c_2^1 + c_3^1$, $c_5'' = c_2'' + c_3''$,

$$c_3^1 = \left[\frac{V(L + L^*)\{1 + Kq - (K - 1)x_P(P/V)\}}{k_L}\right] \quad (11)$$

and

$$c_3'' = \left[\frac{V(L + L^*)(K - 1)\{(L/V) - q\}}{k_L}\right]$$

The solution of Eq. (10) is represented by

$$Z = \left[\frac{2c_5^1 + \varphi c_5^{\prime\prime}}{c_1\,\Delta(\psi)}\right] \tanh^{-1}\left[\frac{(x_Z - x_0)\,\Delta(\psi)}{(x_Z + x_0)\varphi - 2x_Z x_0 - 2x_P\psi K}\right] + - \frac{c_5^{\prime\prime}}{2c_1} \ln\left[\frac{|x_Z^2 - \varphi x_Z + \psi K x_P|}{|x_0^2 - \varphi x_0 + \psi K x_P|}\right] \quad (10a)$$

where

$$\psi = P/c_1$$

$$\Delta(\psi) = [\{1 + \psi + (K-1)\psi x_P\}^2 - 4K\psi x_P]^{1/2} \quad (12)$$

$$\varphi = 1 + \psi + (K-1)\psi x_P$$

and x_Z and x_0 represent the values of x, where $z = Z$ and $z = 0$; at total reflux, $P = 0$.

Then Eq. (10) may be written as

$$\frac{(c_5^1 + x c_5^{\prime\prime})\,dx}{x(1-x)} = c_1\,dz \quad (10b)$$

The solution of this equation is

$$Z = \frac{c_5^1}{c_1} \ln\left\{\frac{x_Z(1-x_0)}{x_0(1-x_Z)}\right\} + \frac{c_5^{\prime\prime}}{c_1} \ln\frac{(1-x_0)}{(1-x_Z)} \quad (10c)$$

The column equation has been derived for a long circular channel. As indicated (2), this column equation may also be applied to packed columns when the right values are given to the parameters concerned. The various parameters of the column equation cannot be calculated because of a number of reasons previously mentioned (2). However, the parameters may be determined by experimental means. In the literature the concepts of theoretical plate and transfer unit are frequently used to express in numerical terms the efficiency of a column. Furthermore, these concepts are applied to indicate the difficulty of a distinct separation. The limits and the possibilities of these two concepts (which have been somewhat unclear until now) can be deduced from suitable relations between these concepts and the parameters governing the separation process. These relations may be obtained from the column equation, as will be shown in the following sections.

HEIGHT EQUIVALENT TO ONE THEORETICAL PLATE

Today the theoretical plate is commonly defined as follows: When the mole fractions of the ascending phase at the upper side and those of the descending phase at the lower side of a section of the column are related to each other, *as if they were at equilibrium,* the length of such a section equals the length of a theoretical plate (*4,5*). This length is often indicated as HETP (height equivalent to one theoretical plate).

The general relation between HETP and the column equation according to Eq. (10a) is rather complicated. However, this relation reduces to a very simple one at total reflux, when $P = 0$. This relation at total reflux is very important because the determination of HETP usually occurs at total reflux.

According to the definition of the theoretical plate,

$$K = \left(\frac{x_z}{1 - x_z}\right)\left(\frac{1 - y_{z-h}}{y_{z-h}}\right) \tag{13}$$

When $P = 0$:

$$K = \left(\frac{x_z}{1 - x_z}\right)\left(\frac{1 - x_{z-h_0}}{x_{z-h_0}}\right) \tag{13a}$$

where h is HETP (centimeters) and h_0 is HETP at total reflux (centimeters).

According to Eq. (10c),

$$h_0 = \frac{c_5^1}{c_1} \ln K + \frac{c_5''}{c_1} \ln \left\{\frac{1 - x_{z-h_0}}{1 - x_z}\right\} \tag{14}$$

Equation (14) shows that HETP at total reflux depends on a number of physical and chemical parameters, determined both by the system applied and the exchange apparatus used. It even depends to some extent on the mole fraction x. HETP may only be assumed to be independent of the mole fraction x when the factor

$$c_5'' \ln \left(\frac{1 - x_{z-h_0}}{1 - x_z}\right)$$

may be neglected with respect to $c_5^1 \ln K$.

To consider when this approximation is permissible, the ratio between

$$c_5'' \ln \left(\frac{1 - x_{z-h_0}}{1 - x_z}\right)$$

and $c_5^1 \ln K$ has been calculated for the case in which $c_2'' \ll c_2^1$ and $c_2^1 = A[V(L + L^*)/k_L]$. Then:

$$\frac{c_5'' \ln (1-x_{z-h_0})/(1-x_z)}{c_5^1 \ln K} = \frac{(K-1)(1-q) \ln (1-x_{z-h_0})/(1-x_z)}{(A+1+Kq) \ln K} = \frac{h_0''}{h_0^1} \quad (15)$$

and

$$\frac{h_0}{h_0^1} = \frac{h_0^1 + h_0''}{h_0^1} \quad (16)$$

The ratio between h_0 and h_0^1 has been calculated for several values of K, q, c_2^1/c_3^1, and x. Some results are summarized in Table 1. The results of Table 1 show to which extent HETP may depend on the mole fraction x for the case in which $c_2'' \ll c_2^1$ and $c_2^1 = A[V(L+L^*)/k_L]$. It appears from Table 1 that for a given value of the equilibrium constant, the relation between plate length and mole fraction may strongly depend on the values of the (often unknown) parameters. From Table 1 it may be concluded that the x dependence of HETP will be very small when the equilibrium constant does not deviate far from unity or when the concentration of A is small with respect to that of B. Then HETP provides a suitable unit of column length, giving the efficiency of the column when using a distinct system.

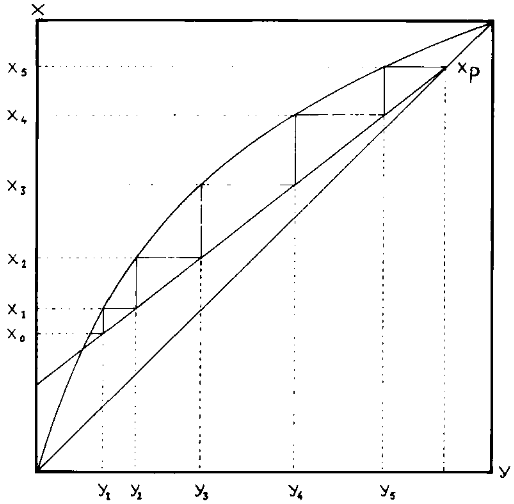

FIG. 3. McCabe-Thiele procedure.

TABLE 1

Dependence of HETP on the Mole Fraction

	$h_0/h_0^{1,a,b}$														
	$x = 0.1$					$x = 0.5$					$x = 0.9$				
		$q = 0.1$		$q = 10$			$q = 0.1$		$q = 10$			$q = 0.1$		$q = 10$	
K	$q = 1$	$A = 0^c$	$A = 10$	$A = 0$	$A = 10$	$q = 1$	$A = 0$	$A = 10$	$A = 0$	$A = 10$	$q = 1$	$A = 0$	$A = 10$	$A = 0$	$A = 10$
1.1	1	1.01	1.001	0.99	0.996	1	1.04	1.004	0.96	0.98	1	1.07	1.01	0.93	0.96
1.5	1	1.03	1.003	0.98	0.99	1	1.17	1.02	0.87	0.92	1	1.34	1.04	0.75	0.85
2.0	1	1.05	1.006	0.97	0.98	1	1.31	1.04	0.82	0.88	1	1.65	1.07	0.63	0.75
3.0	1	1.09	1.01	0.96	0.97	1	1.51	1.06	0.79	0.84	1	2.15	1.13	0.52	0.63

[a] h_0 = HETP at total reflux.

[b] h_0^1 = HETP at total reflux, when neglecting the x-dependent term of Eq. (14) (it is assumed that $c_2'' \ll c_2^1$).

[c] $A = \dfrac{c_2^1}{\{V(L+L^*)/k_L\}}$

When K approaches unity, Eq (14) may be reduced to

$$h_0 = \frac{c_5^1}{L} \text{ (at total reflux, } K \text{ approaching unity)} \qquad \text{(14a)}$$

As is known, the number of theoretical plates required at a distinct withdrawal rate fraction P/L to produce a product of composition x_P from a starting material of composition x_0 is commonly determined by counting the numbers of steps between equilibrium curve and operating line, according to the McCabe-Thiele method (5,6), as indicated in Fig. 3. When the McCabe-Thiele diagram according to Fig. 3 is used, it is supposed that, correspondingly, a concentration distribution occurs in the column, as depicted by Fig. 4.

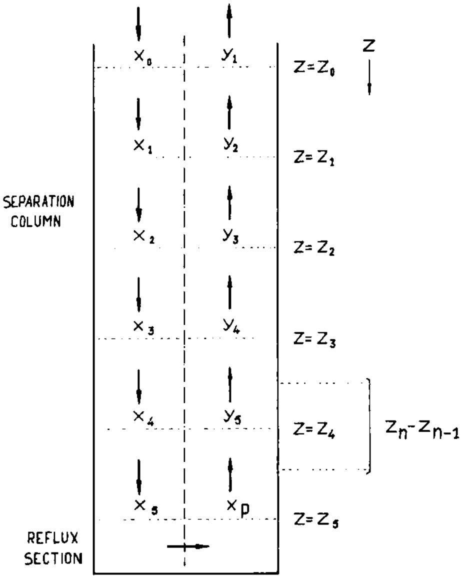

FIG. 4. Concentration distribution in a separation column corresponding to the McCabe-Thiele diagram, according to Fig. 3.

The McCabe-Thiele procedure is based on the tacit assumption that HETP does not depend on the withdrawal rate fraction P/L. This assumption can easily be checked as follows. By introducing the approximation

$$z_n - z_{n-1} = \left(\frac{dz}{dx}\right)_{x=x_n} (x_n - x_{n-1}) = \text{HETP} \qquad (17)$$

where n refers to the position in the column (see Fig. 4), the difference $z_n - z_{n-1}$ may be simply calculated. The factor dz/dx may be obtained from Eq. (10) and the difference $x_n - x_{n-1}$ from the equilibrium condition according to Eq. (5) and the mass balance according to $Lx - Vy = Px_P$. The ratio between $(z_n - z_{n-1})_{P/L=0.1}$ and $(z_n - z_{n-1})_{P=0}$ has been calculated for $K = 2$, $x_P = 0.9$, and some values of x. The results are summarized in Table 2. The dependence of c_5^1 and $c_5^{l'}$ on P/L has been neglected. It follows from Table 2 that the concentration gradient dx/dz and the mole fraction difference $x_n - x_{n-1}$ vary strongly in dependence on the withdrawal rate fraction P/L. However, HETP remains constant. Therefore the assumption that HETP is not a function of the production rate, tacitly introduced in the McCabe-Thiele procedure, appears to be permissible within wide limits, provided that the dependence of c_5^1 and $c_5^{l'}$ on P/L may be neglected.

TABLE 2

Dependence of HETP on the Withdrawal Rate[a]

x	$\frac{(dz/dx)_{P/L=0.1}}{(dz/dx)_{P=0}}$	$\frac{(x_n - x_{n-1})_{P/L=0.1}}{(x_n - x_{n-1})_{P=0}}$	$\frac{(z_n - z_{n-1})_{P/L=0.1}}{(z_n - z_{n-1})_{P=0}}$
0.2	8.89	0.113	1.00
0.5	1.51	0.66	1.00
0.8	1.21	0.82	1.00

[a] $K = 2$ and $x_P = 0.9$.

HEIGHT EQUIVALENT TO ONE TRANSFER UNIT

The concept of the transfer unit (7) is based on the assumption that the interfacial mass transfer F_A is proportional to $x^* - x$ or $y - y^*$ according to:

$$F_A = k_L^*(x^* - x) = k_V^*(y - y^*) \tag{18}$$

where k_L^* and k_V^* are the coefficients of mass transfer, as defined by Eq. (18).

The ratio of the accomplished change in concentration to the driving force $x^* - x$ may be simply related to the height of a transfer unit HTU$_L$ as follows:

$$\frac{dx}{x^* - x} = \frac{dz}{\text{HTU}_L} \tag{19}$$

where HTU_L is the transfer unit belonging to $x^* - x$, the driving force in the descending phase.

Introduction of Eq. (7) into (19) followed by combination with the column equation according to Eq. (10) leads to

$$\mathrm{HTU}_L = \frac{c_5^1 + xc_5^{\prime\prime}}{V\{1 + (K-1)y\}} \tag{20}$$

Equation (20) indicates in which way HTU_L is related to the various parameters appearing in the column equation. When $P = 0$, $c_2^{\prime\prime} \ll c_2^1$, and $c_2^1 = [AV(L + L^*)]/k_L$,

$$\mathrm{HTU}_L = \left[\frac{A + 1 + Kq + x(K-1)(1-q)}{1 + (K-1)x}\right] \frac{L + L^*}{k_L} \tag{20a}$$

When assuming

$$\mathrm{HTU}_L^1 = (A + 1 + Kq)\,\frac{L + L^*}{k_L} \tag{20b}$$

the ratio between HTU_L and HTU_L^1 indicates to which extent HTU_L may vary as a function of the mole fraction x.

In the same way the following may be deduced:

$$\frac{\mathrm{HTU}_V}{\mathrm{HTU}_V^1} = \left[\frac{A + 1 + Kq + x(K-1)(1-q)}{K - (K-1)x}\right]\left[\frac{K}{A + 1 + Kq}\right] \tag{21}$$

The ratios between HTU_L and HTU_L^1 and between HTU_V and HTU_V^1 have been calculated for several values of K, q, c_2^1/c_3^1, and x. The results are summarized in Table 3. Table 3 shows that for a given value of the equilibrium constant the dependence of HTU_L and HTU_V on x may vary considerably as a function of q and A. When K approaches unity or when the value of x is small, the dependence of HTU_L and HTU_V on x may be neglected. Then the transfer unit will be a satisfactory unit of column length. When K approaches unity and $xc_5^{\prime\prime}$ is small with respect to c_5^1, Eq. (20) may be reduced to

$$\mathrm{HTU}_L = \frac{c_5^1}{L} = \mathrm{HETP} \text{ (total reflux, } K \text{ approaching unity)} \tag{20c}$$

It follows from Eq. (20c) that the length of a transfer unit equals the length of a theoretical plate when the equilibrium constant K approaches unity.

TABLE 3

Dependence of HTU on the Mole Fraction

	$\mathrm{HTU}_L/\mathrm{HTU}_L^1$ [a]															$\mathrm{HTU}_V/\mathrm{HTU}_V^1$ [b], $x = 0.9$				
	$x = 0.1$					$x = 0.5$					$x = 0.9$									
		$q = 0.1$		$q = 10$			$q = 0.1$		$q = 10$			$q = 0.1$		$q = 10$			$q = 0.1$		$q = 10$	
K	$q = 1$	$A = 0$ [c]	$A = 10$	$A = 0$	$A = 10$	$q = 1$	$A = 0$	$A = 10$	$A = 0$	$A = 10$	$q = 1$	$A = 0$	$A = 10$	$A = 0$	$A = 10$	$q = 1$	$A = 0$	$A = 10$	$A = 0$	$A = 10$
1.1	0.99	1.0	0.99	0.98	0.99	0.95	0.99	0.96	0.92	0.93	0.92	0.98	0.92	0.86	0.88	1.09	1.17	1.10	1.02	1.05
1.5	0.95	0.99	0.96	0.93	0.94	0.80	0.96	0.82	0.69	0.73	0.69	0.93	0.71	0.51	0.58	1.43	1.93	1.48	1.07	1.21
2.0	0.91	0.98	0.92	0.87	0.88	0.67	0.92	0.69	0.52	0.57	0.53	0.88	0.56	0.32	0.39	1.82	3.04	1.95	1.12	1.34
3.0	0.83	0.95	0.85	0.78	0.80	0.50	0.85	0.54	0.35	0.39	0.36	0.80	0.41	0.17	0.22	2.50	5.61	2.86	1.20	1.52

[a] $\frac{\mathrm{HTU}_L}{\mathrm{HTU}_L^1} = \left[\frac{A + 1 + Kq + x(K-1)(1-q)}{1 + (K-1)x}\right]\left[\frac{1}{A + 1 + Kq}\right]$

[b] $\frac{\mathrm{HTU}_V}{\mathrm{HTU}_V^1} = \left[\frac{A + 1 + Kq + x(K-1)(1-q)}{K - (K-1)x}\right]\left[\frac{K}{A + 1 + Kq}\right]$

[c] $A = \frac{c_2^1}{\left[\frac{V(L + L^\circ)}{k_L}\right]}$

DILUTE MIXTURES AND EXTRACTION AND ABSORPTION PROCESSES

When the concentration of component *A* is small with respect to that of component *B*, the equilibrium condition according to Eq. (5) may be reduced to

$$K = x^1/y^1 \tag{22}$$

From this reduced equilibrium condition, a simplified column equation may be derived as follows:

$$(x^1 - x) = \frac{Ky - x}{1 + Kq} = \frac{(x^* - x)}{1 + Kq} \tag{23}$$

$$F_A = k_L \left\{\frac{Ky - x}{1 + Kq}\right\} = (L + L^*)\frac{\delta x}{\delta z} + c_{6,L}\frac{\delta x}{\delta t} \tag{24}$$

Combination of Eqs. (4) and (24) leads to

$$(c_5 + c_2''x)\frac{dx}{dz} = (c_1 + P)x - PKx_P \tag{25}$$

where $c_5 = c_2^1 + c_3$ and $c_3 = \{[V(L + L^*)(1 + Kq)]/k_L\}$.

When dealing with extraction and absorption processes, the distribution of one compound *A* between two phases is considered. The equilibrium condition of such a distribution may be represented by Eq. (22). This means that the column equation according to Eq. (25) may also be applied to the description of extraction and absorption processes, provided that L, V, c_L, c_V, D_L, and D_V are constant throughout the column. This condition is fulfilled only when the concentrations of A in both phases are sufficiently small. Large concentrations of *A* may be considered by introducing the variations of flow rates and concentrations into the mass balances used in the derivation of the column equation. When dealing with a column not connected with a reflux section, as often will be the case in extraction and absorption processes, the parameter x_P occurring in the column equation stands for an imaginary concentration not occurring in the column. Then the factor x_P may be related to the top and bottom concentrations of the column according to

$$Px_P = Lx_Z - Vy_Z = Lx_0 - Vy_0 \tag{26}$$

The dependence of HETP and HTU on the mole fraction when dealing with extraction and absorption processes will be considered by relating the two units with the simplified column equation according to Eq. (25). When $h = \text{HETP}$,

$$K = \frac{x_z}{y_{z-h}} \tag{27}$$

Introduction of relation (27) into Eq. (25) leads to

$$h = \left[\frac{c_5}{(LK - V)} + c_2'' \frac{K(Lx_0 - Vy_0)}{(LK - V)^2}\right] \ln\left(\frac{LK}{V}\right) + \frac{c_2''}{(LK - V)}(x_z - x_{z-h}) \tag{28}$$

In many cases c_2'' will be small with respect to c_5. Then Eq. (28) may be reduced to

$$h = \frac{c_5}{(LK - V)} \ln\left(\frac{LK}{V}\right) \tag{28a}$$

When LK/V approaches unity,

$$h = \frac{c_5}{V} \tag{28b}$$

Relation of the concept of HTU to the simplified column equation leads to

$$\text{HTU}_L = \frac{c_5 + xc_2''}{V} \tag{29}$$

Generally xc_2'' will be small with respect to c_5. Then, according to the Eqs. (28b) and (29), the dependence of both HTU_L and HETP on the concentration may be neglected.

Both HTU_L and HETP appear to be suitable units of column length when the concentration of one component is small with respect to the other. HTU and HETP depend somewhat differently on the equilibrium constant and the flow rates, as follows from the Eqs. (28a) and (29).

DISCUSSION

By the use of a more general expression for the interfacial mass transfer, it appeared possible to extend the applicability of the column equation to values of the equilibrium constant up to 3. The parameters of the column equation cannot be calculated because of a number of reasons mentioned previously (2). The parameters may be determined by means of suitable experiments. The determination of the parameters of the column equation will be of importance when investigating the conditions for obtaining optimal separations and when calculating optimal production rates.

For reasons of simplicity, the concepts of theoretical plate and transfer unit are frequently used in books and papers dealing with separation processes. In the foregoing sections these two concepts have been related to the parameters occurring in the column equation. It appears from these relations that both the theoretical plate and the transfer unit are satisfactory units of column length, provided that the compounds are difficult to separate or that the concentration of one compound is small with respect to the other.

The more the equilibrium constant K deviates from unity, the more HETP and HTU become dependent on the mole fraction. Then HETP and HTU are not constant throughout the column.

The regions of applicability of HETP and HTU may be somewhat different when dealing with exchange processes with K deviating from unity (e.g., > 1.5). Depending on the parameters of the column equation, HETP may be preferable to HTU or vice versa. Some cases, in which one of the units is preferable to the other are indicated in Table 4. These data have been deduced from the Tables 1 and 3.

TABLE 4

Choice of Unit of Column Length when Dealing with Exchange Processes with $K > 1.5$

Parameters column equation	Preferable unit
$q = 1$	HETP
$q = 0.1, A = 0$	HTU_L
$q = 0.1, A = 10$	HETP
$q = 10, A = 0$	HTU_V
$q = 10, A = 10$	HETP, HTU_V
Unknown	HETP

When the parameters of the column equation are not known, the application of HETP is recommended because the dependence of HETP on the mole fraction generally is smaller than that of HTU.

Because HTU is based on a differential approach of the concentration distribution in the column, HTU may be easily related to the differential form of the column equation, as was shown. On the other hand, HETP must be related to the integrated form of the column equation, which leads to a rather complicated relation. Fortunately, this complicated relation may be reduced to a simple one in a number of important cases.

An additional advantage of the concept of HETP is the fact that this unit refers to a clear task of separation. This is not the case when dealing with the concept of HTU; then two units must be distinguished, HTU_L and HTU_V, referring to supposed driving forces in the descending and ascending phases.

It is sometimes supposed (*6,8*) that the use of the theoretical plate concept is basically unsound because it would involve a stepwise change from one plate to the next one, whereas the change in composition in a packed column is gradual and continuous from top to bottom. This opinion does not hold when the definition of HETP given above is considered. Summarizing, it may be said that the assumption of many authors (*6,8–10*) that the concept of the transfer unit is more satisfactory than the concept of the theoretical plate when dealing with packed columns is not confirmed by the relations derived. The calculations presented show that in several cases HETP may be a more suitable unit of column length than HTU.

List of Symbols

c_1, c_2^1, $c_2^{\prime\prime\prime}$, c_3, c_3^1, $c_3^{\prime\prime\prime}$, c_5, c_5^1, $c_5^{\prime\prime\prime}$, c_6^1, $c_6^{\prime\prime\prime}$	column parameters
c_L, c_V	sum of the concentrations of components *A* and *B* in the descending and ascending phases (moles per milliliter)
D_L, D_V	axial dispersion coefficient in the descending and ascending phases (square centimeters per second)
F_A	net interfacial mass transfer of component *A* in a column of unit length (moles per second-centimeter)
h	HETP (centimeters)
h_0	HETP at total reflux (centimeters)

HTU_L	transfer unit belonging to $x^* - x$, driving force in the descending phase (centimeters)
HTU_V	transfer unit belonging to $y - y^*$, driving force in the ascending phase (centimeters)
k_L, k_V	coefficients of mass transfer in the descending and ascending phases (moles per second-centimeter)
k_L^*, k_V^*	assumed coefficients of mass transfer when dealing with the concept of the transfer unit (moles per second-centimeter)
K	equilibrium constant (it is assumed that $K > 1$)
L	flow rate in the descending phase (moles per second)
L^*	corrections due to the axial dispersion (moles per second)
P	product flow (moles per second)
q	k_L/k_V
t	time (seconds)
V	flow rate in the ascending phase (moles per second)
x, y	flow rate fractions of component A in the descending and ascending phases
x^1, y^1	flow rate fractions of component A at the interface
x^*, y^*	represent compositions of the descending and ascending phases, which would be in equilibrium with the bulk compositions of the ascending and descending phases of a cross section of the column
x_P	flow rate fraction of component A in the product flow
x_Z, x_0	values of x, where $z = Z$ and $z = 0$
z	coordinate in the longitudinal direction (centimeters); z increases from $z = 0$ at the inlet up to $z = Z$ at the beginning of the reflux section

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